Tarski's theory of truth: An example

Consider the model $\mathcal{M} = \langle \mathbf{D}, I \rangle$. We compute the denotation of the formula associated with (1a) **in a model where there are two olives and Jack likes only one of them**, so that in the end we should get to $[\![\varphi]\!]^g_{\mathcal{M}} = 0$ for any assignment function g.

(1) a. Jack likes olives.
  b. $\varphi : \forall x\, (Ox \rightarrow Ljx)$

First, we can translate (1a) into predicate logic as in (1b), where O is the (unary) predicate for objects that are olives, the free variable j represents Jack and L is the (binary) predicate for the transitive verb 'to like'.

Let us start by computing some basic denotations:

$$[\![x]\!]^g_{\mathcal{M}} = g(x) \in \mathbf{D} \tag{2}$$
$$[\![j]\!]^g_{\mathcal{M}} = I(j) = \mathbf{j} \in \mathbf{D} \tag{3}$$
$$[\![O]\!]^g_{\mathcal{M}} = \{\mathbf{o}_1, \mathbf{o}_2\} \subseteq \mathbf{D} \tag{4}$$
$$[\![L]\!]^g_{\mathcal{M}} = \{(\mathbf{j}, \mathbf{o}_1)\}, \tag{5}$$

regardless of the assignment function g. Note that we used I for the free variable (*constant*) j and g for the variable x in (1b).

Recall that

$$[\![\phi \rightarrow \psi]\!]^g_{\mathcal{M}} = 1 \text{ iff } [\![\phi]\!]^g_{\mathcal{M}} = 0 \text{ or } [\![\psi]\!]^g_{\mathcal{M}} = 1 \tag{6}$$
$$[\![\forall y\ \psi]\!]^g_{\mathcal{M}} = 1 \text{ iff for all } \mathbf{d} \in \mathbf{D}, [\![\psi]\!]^{g[y/\mathbf{d}]}_{\mathcal{M}} = 1 \tag{7}$$

Thus, using (7), we shall focus on

$$[\![Ox \rightarrow Ljx]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}},$$

where $\mathbf{d} \in \mathbf{D}$, especially $\mathbf{d} \in \{\mathbf{o}_1, \mathbf{o}_2\}$ as we will see in a moment.

On the other hand, using (6),

$$[\![Ox \rightarrow Ljx]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 1 \text{ iff } [\![Ox]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 0 \text{ or } [\![Ljx]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 1$$

Conversely,

$$[\![Ox \rightarrow Ljx]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 0 \text{ iff } [\![Ox]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 1 \text{ and } [\![Ljx]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 0$$

<u>*General* case:</u> $\mathbf{d} \in \mathbf{D} \setminus \{\mathbf{o}_1, \mathbf{o}_2\}$, including $\mathbf{d} = \mathbf{j}$

Suppose $\mathbf{d} \in \mathbf{D} \setminus \{\mathbf{o}_1, \mathbf{o}_2\}$. Using (2) and (4), we have

$$[\![x]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = g[x/\mathbf{d}](x) = \mathbf{d}$$
$$[\![O]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = \{\mathbf{o}_1, \mathbf{o}_2\}$$

Since $\mathbf{d} \notin \{\mathbf{o}_1, \mathbf{o}_2\}$, it follows that $[\![Ox]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 0$, hence $[\![Ox \rightarrow Ljx]\!]^{g[x/\mathbf{d}]}_{\mathcal{M}} = 1$.

<u>First olive case:</u> $\mathbf{d} = \mathbf{o}_1$

For $\mathbf{d} = \mathbf{o}_1$, using (2), (3), (4) and (5), we have

$$[\![x]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}} = \mathbf{o}_1 \in \{\mathbf{o}_1, \mathbf{o}_2\} = [\![O]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}}$$
$$\left([\![j]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}}, [\![x]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}}\right) = (\mathbf{j}, \mathbf{o}_1) \in \{(\mathbf{j}, \mathbf{o}_1)\} = [\![L]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}}$$

This way, although $[\![Ox]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}} = 1$, we have $[\![Ljx]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}} = 1$, so $[\![Ox \rightarrow Ljx]\!]^{g[x/\mathbf{o}_1]}_{\mathcal{M}} = 1$.

Second olive case: $\mathbf{d} = \mathbf{o}_2$

For $\mathbf{d} = \mathbf{o}_2$, using (2), (3) and (5), we have

$$\left([\![j]\!]_{\mathcal{M}}^{g[x/\mathbf{o}_2]}, [\![x]\!]_{\mathcal{M}}^{g[x/\mathbf{o}_2]} \right) = (\mathbf{j}, \mathbf{o}_2) \notin \{(\mathbf{j}, \mathbf{o}_1)\} = [\![L]\!]_{\mathcal{M}}^{g[x/\mathbf{o}_2]}$$

Hence, we both have $[\![Ox]\!]_{\mathcal{M}}^{g[x/\mathbf{o}_2]} = 1$ and $[\![Ljx]\!]_{\mathcal{M}}^{g[x/\mathbf{o}_2]} = 0$, so $[\![Ox \rightarrow Ljx]\!]_{\mathcal{M}}^{g[x/\mathbf{o}_2]} = 0$.

As we see, the 'general' case proves to be less *interesting* since the antecedent of the conditional in (1b) is trivially false for any non-olive element of the domain. That is why we 'shortcut' the computation of the denotation by focusing only on the olive elements of the domain.

Given that (1b) comprises a universal quantifier, we need to show that the denotation is 1 for all elements of the domain (as we did). However, if we 'know' (or have the intuition) that the sentence is not true (denotation equal to 0), it **suffices** to verify that the denotation of the sub-formula is 0 for a specific element of the domain, here $\mathbf{d} = \mathbf{o}_2$.