HW3 - First-order logic (Solution)

April 6, 2025

1. *[3 pts.]* For each of the following predicate logic formulae, number the (different) occurrences of the variables with subscripts according to their scope. For instance,

$$\exists x\ (\exists y\ Axyz \rightarrow Bxyz)$$

would become

$$\exists x_1\ (\exists y_1\ Ax_1y_1z_1 \rightarrow Bx_1y_2z_1).$$

(a) $\exists x\ (Axy \vee By)$

Solution: $\exists x_1\ (Ax_1y_1 \vee By_1)$

(b) $\exists x\ (Axx \wedge \forall x\ Bxy) \vee \exists y\ Cy$

Solution: $\exists x_1\ (Ax_1x_1 \wedge \forall x_2\ Bx_2y_1) \vee \exists y_2\ Cy_2$

(c) $\exists x\ (\exists y\ Axy \vee By)$

Solution: $\exists x_1\ (\exists y_1\ Ax_1y_1 \vee By_2)$

(d) $\forall x \forall y\ ((Axy \wedge By) \rightarrow \exists w\ Cxw)$

Solution: $\forall x_1 \forall y_1\ ((Ax_1y_1 \wedge By_1) \rightarrow \exists w_1\ Cx_1w_1)$

(e) $\forall x\ (\forall y\ Azx \rightarrow By)$

Solution: $\forall x_1\ (\forall y_1\ Az_1x_1 \rightarrow By_2)$

(f) $\forall x \forall y\ Ayy \rightarrow Bx$

Solution: $\forall x_1 \forall y_1\ Ay_1y_1 \rightarrow Bx_2$

2. *[3 pts.]* Simplify the following expression:

$$(q \rightarrow (p \wedge r)) \rightarrow (p \rightarrow q)$$

Solution:

$$\begin{aligned}
&(q \rightarrow (p \wedge r)) \rightarrow (p \rightarrow q) && \\
&\equiv (\neg q \vee (p \wedge r)) \rightarrow (p \rightarrow q) && \text{[material implication]}\\
&\equiv (q \wedge (\neg p \vee \neg r)) \vee (p \rightarrow q) && \text{[material implication, De Morgan's]}\\
&\equiv (q \vee (p \rightarrow q)) \wedge (\neg p \vee \neg r \vee (p \rightarrow q)) && \text{[distribution]}\\
&\equiv (\neg p \vee q) \wedge (\neg p \vee \neg r \vee q) && \text{[material implication, idempotence]}\\
&\equiv \neg p \vee q \vee (\bot \wedge \neg r) && \text{[identity, distribution]}\\
&\equiv \neg p \vee q \vee \bot && \text{[domination]}\\
&\equiv \neg p \vee q && \text{[identity]}\\
&\equiv (p \rightarrow q) &&
\end{aligned}$$

3. *[8 pts.]* Translate as precisely as possible the following sentences into predicate logic. If ambiguous, provide a formula for each possible reading.

(1)
a. All superheroes had a difficult time.
b. Every time Mia finds a wallet she gives it back to its owner.
c. All newspapers which don't have readers will disappear if they don't find a buyer.
d. Every student who solves a problem will explain it.
e. There are only two solutions to every problem.
f. Everyone is marked by an unrequited love.

g. Nobody who is not a French likes cheese better than any French.
h. Only completely consistent people are dead.

Solution:

(a) $\forall x\ (Sx \to Dx)$, where S is the predicate for 'superhero', and D for 'having a difficult time'.

(b) $\forall x\ ((Wx \wedge Fmx) \to \Box)$, where $\Box$ is

- $\exists y\ (Oyx \wedge Gmxy)$ if we translate *its owner* as *an owner*
- $\forall y\ (Oyx \to Gmxy)$ if we translate *its owner* as *any owner*
- $\exists y\ (Oyx \wedge \forall z\ (Ozx \to z = y) \wedge Gmxy)$ if we translate *its owner* as *the owner*

Here W is the predicate for 'wallet', Fab for 'a finds b', Oab for 'a owns b', $Gabc$ for 'a gives b to c', and m represents Mia.

(c) $\forall x\ ((Nx \wedge \neg\exists y(Py \wedge Ryx) \wedge \neg\exists z\ (Bz \wedge Fxz)) \to Dx)$, if each newspaper has to find its own buyer.

$\exists z(Bz \wedge \forall x\ ((Nx \wedge \neg\exists y(Py \wedge Ryx) \wedge \neg Fxz) \to Dx))$, if there is a single buyer for all newspapers.

(d) $\forall x \forall y\ ((Sx \wedge Py \wedge Rxy) \to Exy)$, if 'student' has a larger scope than 'problem'.

$\exists y\ (Py \wedge \forall x\ ((Sx \wedge Rxy) \to Exy))$, if 'problem' has a larger scope than 'student'.

(e) $\forall x\ (Px \to \exists y \exists z\ (Syx \wedge Szx \wedge y \neq z \wedge \forall w\ (Swx \to (w = y \vee w = z))))$, if each problem has its own two solutions.

$\exists y \exists z\ (y \neq z \wedge \forall x\ (Px \to (Syx \wedge Szx \wedge \forall w\ (Swx \to (w = y \vee w = z)))))$, if all problems have the same two solutions.

(f) $\forall x\ (Px \to \exists y\ (Py \wedge Lxy \wedge \neg Lyx \wedge Mxy))$ for a narrow scope.

$\exists y\ (Py \wedge \forall x\ (Px \to (Lxy \wedge \neg Lyx \wedge Mxy))$ for a broad scope.

(g) $\neg\exists x\ (\neg Fx \wedge \exists y\ (Fy \wedge Cxy)$

(h) $\forall x\ (Dx \to (Px \wedge Cx))$

4. *[3 pts.]* Consider a language that includes binary predicates P and equality ($=$). Consider the three following formulae:

$(F_1)\ \forall x \forall y\ (Pxy \to x \neq y)$ $\qquad (F_2)\ \forall x \forall y\ (Pxy \wedge x \neq y)$ $\qquad (F_3)\ \forall x \exists y\ (Pxy \wedge x \neq y)$

For each formula, determine whether there exists a model $\mathcal{M} = \langle D, I \rangle$ that satisfies it:

(a) when D is a singleton,

(b) when D contains exactly two elements,

(c) when $D = \mathbb{N} = \{0, 1, 2, \dots\}$ and $I(P) = \{(x, y) \mid x \text{ is divisible by } y\}$.

Solution:

(a) A singleton $D = \{a\}$:

- F_1 reduces to $Paa \to a \neq a$, so it suffices to have $[\![Paa]\!]^g_{\mathcal{M}} = 0$ for F_1 to be satisfied. **This is easily achieved when $I(P) = \emptyset$**: $(a, a) \notin I(P) = \emptyset$.
- F_2 reduces to $Paa \wedge a \neq a$. The second term of the conjunction is always false (regardless of the interpretation of predicate P), so **there does not exist a model $\mathcal{M}$ in which F_2 is satisfied**.
- Recall that the universal (resp. existential) quantifier is equivalent to the conjunction (resp. disjunction) of the quantified formula over all the domain. In the case of a singleton domain, F_3 is also equivalent to $Paa \wedge a \neq a$, so similarly to F_2, **there does not exist such a model**.

(b) $D = \{a, b\}$ contains exactly two elements:

- As in the singleton case, the quantified formula in F_1, $Pxy \to x \neq y$, is satisfied when the antecedent Pxy is not or when both the antecent and consequence are satisfied, and this remains true regardless of the valuations of x and y. Hence, **setting $I(P) = \emptyset$ is sufficient**.
- For F_2 to be satisfied, the quantified formula should be satisfied for any valuation pair $(x, y) \in D^2$. However, as in the singleton case, for the two valuations $x = y = a$ and $x = y = b$, $Pxy \wedge x \neq y$ is not satisfied because the second term of the conjunction is False. Therefore, **there does not exist a model $\mathcal{M}$ in which F_2 is satisfied**.
- Reading F_3, it is apparent that F_3 is satisfied if the interpretation of predicate P contains at least one non-identical pair $(x, y), x \neq y$, for each $x \in D$. In other words, F_3 is satisfied if $I(P) \supseteq \{(a, b), (b, a)\}$. This way, $I(P) = \{(a, b), (b, a)\}$ **works, but $I(P) = \{(a, a), (a, b), (b, a), (b, b)\}$ as well**.

(c) $D = \mathbb{N} = \{0, 1, 2, \dots\}$ and $I(P) = \{(x, y) \mid x \text{ is divisible by } y\}$: **Before delving into the three formulae**, note that for any $x \in D$, $(x, x) \in I(P)$ since any number is divisible by itself.

- F_1 is not satisfied if there is at least one couple $(x, y) \in D^2$ such that $Pxy \to x \neq y$ is not satisfied. As stated in the note above, for any number $x \in D = \mathbb{N}$, $(x, x) \in I(P)$ but $x = x$, so **the quantified formula (and by extension F_1) is not satisfied**.
- We can generalise what we have seen with the previous domains: F_2 **is trivially not satisfied**. Indeed, the quantified formula contains $x \neq y$ and, for F_2 to be satisfied, $x \neq y$ should be true for any $x, y \in D$, but this is definitely not the case for $x = y \in D$.
- **F_3 is not satisfied because of $x = 1$.** Let us show this by contradiction. Suppose there exists some $y \in D$ such that $P1y$ is satisfied and $1 \neq y$. However, 1 is only divisible by itself, so for any $y \in D$, if $(1, y) \in [\![P]\!]^g_{\mathcal{M}}$, then $y = 1$. This is a contradiction, so there does not exist a $y \in D$ such that $P1y \wedge 1 \neq y$ is satisfied.

5. (a) *[1.5 pts.]* Translate the following sentences into predicate logic.

(2) a. John owns everything he has not lost.
b. John has not lost 1 million dollars.
c. John owns one million dollars.

(b) *[1.5 pts.]* Analyse the syllogism going from the conjunction of (2a) and (2b) to the conclusion (2c). Explain where the problem lies.

Solution: The sentences in (2) can be translated into predicate logic as follows:

(3) a. $\forall x\, (\neg Ljx \to Ojx)$
b. $\neg Ljm$
c. Ojm

Interestingly, when formalised this way, the syllogism appears logically valid.

The fundamental issue is that 'not having lost something' is not the same as 'having something'. When we say 'John has not lost 1 million dollars', this statement can be true in two very different scenarios:

(a) John once had 1 million dollars and still has it (didn't lose it)

(b) John never had 1 million dollars in the first place (can't lose what you never had)

The syllogism incorrectly treats these as equivalent. On the other hand, premise (2a) logically applies only to things that John actually possessed at some point: you can only lose something you once had.