Compositionality & λ-calculus

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Overview

Syntax

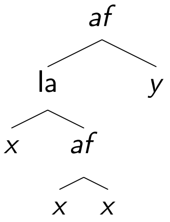
Let V be a countable set of variables. The set of all well-formed terms, Λ, is defined inductively as follows :

- $V \subset \Lambda$
- $\lambda x.t \in \Lambda$ $\quad \forall x \in V, \forall t \in \Lambda$
- $(t_1)t_2 \in \Lambda$ $\quad \forall t_1, t_2 \in \Lambda$

Syntax (cont'd)

The term $(\lambda x.(x)x)y$ has this syntactic structure :

$$af(la(x, af(x,x)), y)$$



Variable substitution

- $x_{[x:=z]} \rightsquigarrow z$
- $y_{[x:=z]} \rightsquigarrow y$ si $y \neq x$
- $(M)N_{[x:=z]} \rightsquigarrow (M_{[x:=z]})N_{[x:=z]}$
- $\lambda x.M_{[x:=z]} \rightsquigarrow \lambda z.M_{[x:=z]}$
- $\lambda y.M_{[x:=z]} \rightsquigarrow \lambda y.M_{[x:=z]}$ if $x \neq y$

Term substitution

- $x_{[x:=t]} \rightsquigarrow t$
- $y_{[x:=t]} \rightsquigarrow y$ si $y \neq x$
- $(M)N_{[x:=t]} \rightsquigarrow (M_{[x:=t]})N_{[x:=t]}$
- $\lambda y.M_{[x:=t]} \rightsquigarrow \lambda y.M_{[x:=t]}$ if y is not free in t.

α equivalence

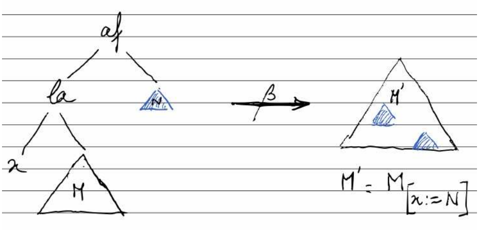
$$\lambda x.\varphi \equiv \lambda z.\varphi_{[x:=z]}$$

Convention on variables

Let M be a term, x a variable. By convention, the occurrences of x in M are either all free or all bound.
It can be shown that every term constructed without respecting this convention is α-equivalent to a term that respects the convention.

β equivalence

$$(\lambda x.M)N \equiv M_{[x:=N]}$$



Combinators

A combinator is a closed λ-term (ie without free variable)

Identity

$\mathsf{I} =_{\mathrm{def}} \lambda x.x$

For any term t : $(\mathsf{I})t \equiv t$

Booleans

$\mathsf{T} =_{\text{def}} \lambda x.\lambda y.x$
$\mathsf{F} =_{\text{def}} \lambda x.\lambda y.y$

This encoding allows to encode an if-then-else function :

`if` P `then` Q `else` $R =_{\text{def}} ((P)Q)R$.

if P is β-equivalent to T then $((P)Q)R$ will yield Q, while if P is β-equivalent (or β-reduces) to F, the outcome will be Q.

The IF combinator

$\text{IF} =_{\text{def}} \lambda b.\lambda t.\lambda f.((b)t)f$

$\text{NOT} =_{\text{def}} \lambda u.((u)\text{F})\text{T}$
$\text{AND} =_{\text{def}} \lambda u.\lambda v.((u)v)\text{F}$
$\text{OR} =_{\text{def}} \lambda u.\lambda v.((u)\text{T})v$

Church numerals

$0 =_{\text{def}} \lambda f.\lambda x.x$

$1 =_{\text{def}} \lambda f.\lambda x.(f)x$

$\mathsf{n} =_{\text{def}} \lambda f.\lambda x.(f)(f)\ldots(f)x$
with n times f

$\mathsf{Succ} =_{\text{def}} \lambda n.\lambda f.\lambda x.(f)((n)f)x$

$+ \equiv \lambda m.\lambda n.\lambda f.\lambda x.((m)f)((n)f)x$

$* \equiv \lambda m.\lambda n.\lambda f.(m)(n)f$

Overview

Frege's principle

The meaning of an expression is uniquely determined by the meanings of its parts and their mode of combination.

Motivation : Humbolt's view on finite means for infinite sentences

Consequences

- Locality principle : lexical items have a meaning that is independant of the expression they occur in.
- Substitution principle : synonymous expressions may be substituted for each other without changing the meaning of the complex expression in which they occur.
- Parts of well formed sentences have « meaning »
- Meanings can be « composed » : Frege's saturation idea

λ-terms can represent individual meanings and functional application can represent semantic composition.

Overview

Type theory

1. e is a type
2. t is a type
3. if a and b are types, then $\langle a, b \rangle$ is a type

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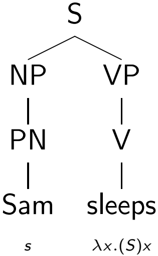
- $D_e = A$
- $D_t = \{0, 1\}$
- $D_{\langle a,b \rangle}$ = the set of mappings from D_a to D_b.

Meaningful expressions

For a, b types :

- variables and individual constants of type a belong to ME_a.
- if $\alpha \in ME_{\langle a,b \rangle}$ and $\beta \in ME_a$ then $(\alpha)\beta \in ME_b$.
- if u is a variable of type a and $\alpha \in ME_b$, then $\lambda u.\alpha \in ME_{\langle a,b \rangle}$.
- if φ and ψ are in ME_t, then the following expressions are also in ME_t : $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$.
- if φ is in ME_t and u is a type a variable, then $\forall u\varphi$ and $\exists u\varphi$ are in ME_t.

Overview



S	$\rightarrow$	NP	VP
$[\![S]\!]$	$\leftarrow$	$([\![VP]\!])$	$[\![NP]\!]$
0	$\leftarrow$	(2)	1
NP	$\rightarrow$	PN	
0	$\leftarrow$	1	
VP	$\rightarrow$	V	
0	$\leftarrow$	1	
PN	$\rightarrow$	Sam	
0	$\leftarrow$	s	
V	$\rightarrow$	sleeps	
0	$\leftarrow$	$\lambda x.(S)x$	

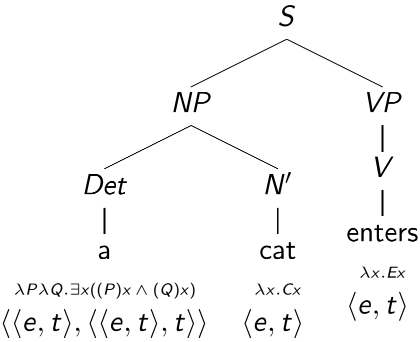
Roadmap for the fragment

- A cat enters — $\exists x\,(Cx \wedge Ex)$
- Sam likes Pam — Lsp (or $((L)s)p$)
- Everyone likes Pam — $\forall x\,(Px \rightarrow Lxp)$
- Everyone likes an actress — $\forall x\,(Px \rightarrow \exists y\,(Ay \wedge Lxy))$
- Sam is mortal — Ms
- Sam met a tall person — $\exists x((Px \wedge Tx) \wedge Msx)$
- Sam doesn't sleep — $\neg Ss$

Quantified sentences

(1) a. A cat enters

b. $\exists x\,(Cx \wedge Ex)$



S	$\rightarrow$	NP	VP
0	$\leftarrow$	(1)	2
NP	$\rightarrow$	Det	N'
0	$\leftarrow$	(1)	2
VP	$\rightarrow$	V	
0	$\leftarrow$	1	

Excursus : Generalized Quantifiers

1. NL Quantifiers vs Logic Quantifiers

- Restriction
- Lack of parallelism

[Jean $_{\text{NP}}$] dort	$\text{dort}(j)$
[Certains hommes $_{\text{NP}}$] dorment	$\exists x\ (Hx \wedge Dx)$
[Tous les hommes $_{\text{NP}}$] dorment	$\forall x\ (Hx \rightarrow Dx)$
[Au moins deux hommes $_{\text{NP}}$] dorment	$\exists x \exists y (x \neq y \wedge Hx \wedge Hy \wedge Dx \wedge Dy)$

- Lack of expressivity

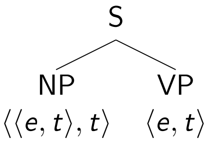
(2) a. Un nombre fini d'étoiles sont sensibles à l'attraction du soleil.
b. Plus de la moitié des amis de Jean sont parisiens.
c. La plupart des gens ont voté Chirac.

Excursus : Generalized Quantifiers

2. Generalized Quantifiers

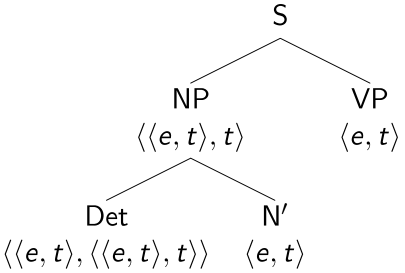
3. Thesis : $[\![\text{NP}]\!] = \text{GQ}$

(3)
a. $[\![\text{Tous les N}]\!] = \{X \subseteq E \ / \ [\![\text{N}]\!] \subseteq X\}$
b. $[\![\text{Quelques N}]\!] = \{X \subseteq E \ / \ [\![\text{N}]\!] \cap X \neq \emptyset\}$
c. $[\![\text{Jean}]\!] = \{X \subseteq E \ / \ j \in X\}$
d. $[\![\text{Au moins deux N}]\!] = \{X \subseteq E \ / \ |[\![\text{N}]\!] \cap X| \geq 2\}$
e. $[\![\text{La plupart des N}]\!] = \{X \subseteq E \ / \ |[\![\text{N}]\!] \cap X| \geq |[\![\text{N}]\!] \setminus X|\}$



Excursus : Generalized Quantifiers

(4)
a. $[\![\text{Tous les } A\ B]\!] = 1 \Leftrightarrow [\![A]\!] \subseteq [\![B]\!]$
b. $[\![\text{Certains } A\ B]\!] = 1 \Leftrightarrow [\![A]\!] \cap [\![B]\!] \neq \emptyset$
c. $[\![\text{La plupart } A\ B]\!] = 1 \Leftrightarrow |[\![A]\!] \cap [\![B]\!]| \geq |[\![A]\!] \setminus [\![B]\!]|$
d. $[\![\text{Beaucoup } A\ B]\!] = 1 \Leftrightarrow |[\![A]\!] \cap [\![B]\!]| \geq m|[\![A]\!]|$



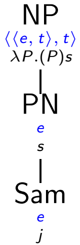
4. Determiners ⊂ binary set relations

Back to Sam

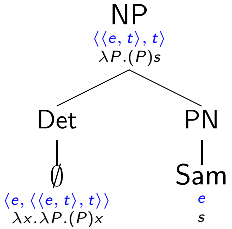
Back to Sam

Back to Sam

Back to Sam



$$SN \quad \rightarrow \quad NP$$
$$0 \quad \leftarrow \quad (\lambda x.\lambda P.(P)x)1$$



Application

$$[\![\text{Sam sleeps}]\!] = \left(\left(\frac{\Phi}{\langle e,\langle et,t\rangle\rangle}\right)\frac{\text{Sam}}{e}\right)\frac{\text{sleeps}}{et}$$

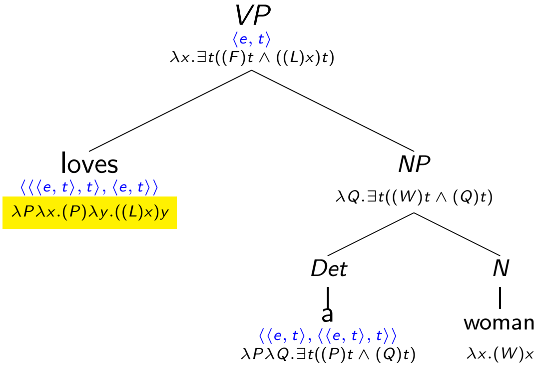
$$= \left(\left(\frac{\lambda u \lambda P.Pu}{\langle e,\langle et,t\rangle\rangle}\right)\frac{s}{e}\right)\frac{\lambda x.Sx}{et}$$

$$= \left(\frac{\lambda P.Ps}{\langle et,t\rangle}\right)\frac{\lambda x.Sx}{et}$$

$$= \left(\frac{\lambda x.Sx}{et}\right)\frac{s}{e}$$

$$= \frac{Ss}{t}$$

Correct version for transitive verbs

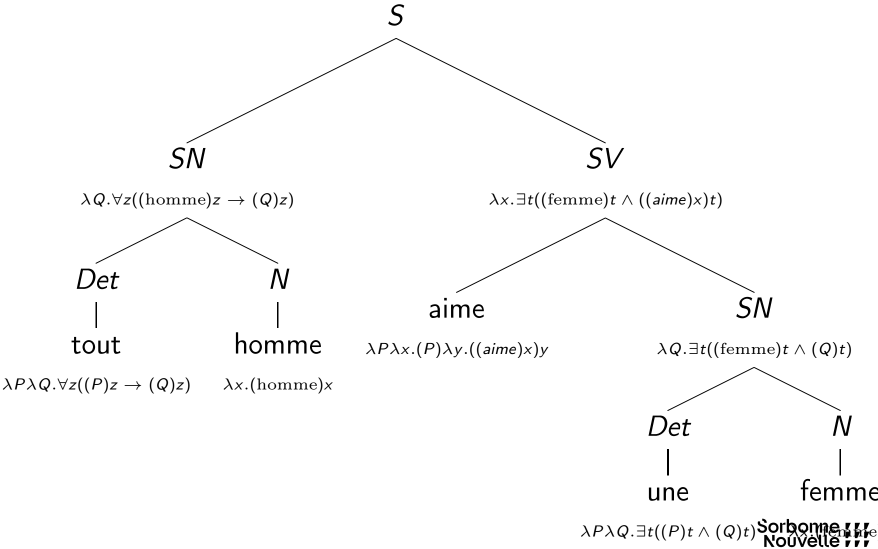


$$\Big(\lambda P \lambda x.(P)\lambda y.((L)x)y\Big)\ \lambda P.(P)m$$

$$\longrightarrow_\beta \quad \lambda x.\ \Big(\lambda P.(P)m\Big)\ \lambda y.((\mathrm{L})x)y$$

$$\longrightarrow_\beta \quad \lambda x.(\lambda y((L)x)y)m$$

$$\longrightarrow_\beta \quad \lambda x.((L)x)m$$


S
SN
$\lambda Q.\forall z((\text{homme})z \to (Q)z)$
SV
$\lambda x.\exists t((\text{femme})t \wedge ((aime)x)t)$
Det
tout
$\lambda P \lambda Q.\forall z((P)z \to (Q)z)$
N
homme
$\lambda x.(\text{homme})x$
aime
$\lambda P \lambda x.(P)\lambda y.((aime)x)y$
SN
$\lambda Q.\exists t((\text{femme})t \wedge (Q)t)$
Det
une
$\lambda P \lambda Q.\exists t((P)t \wedge (Q)t)$
N
femme

Application

$$
\begin{aligned}
[\![\text{Every woman loves a man}]\!] &= ((every)woman)(loves)(a)man \\
&= ((\lambda P \lambda Q.\forall x\ (Px \to Qx))\ \lambda u.Wu)\ (\lambda R \lambda a.(R)\lambda b.Lab)\ (\lambda A \lambda B.\exists y\ (Ay \wedge By))\ \lambda v.Mv \\
&= (\lambda Q.\forall x\ ((\lambda u.Wu)x \to Qx))\ (\lambda R \lambda a.(R)\lambda b.Lab)(\lambda A \lambda B.\exists y\ (Ay \wedge By))\lambda v.Mv \\
&= (\lambda Q.\forall x\ (Wx \to Qx))\ (\lambda R \lambda a.(R)\lambda b.Lab)(\lambda A \lambda B.\exists y\ (Ay \wedge By))\lambda v.Mv \\
&= (\lambda Q.\forall x\ (Wx \to Qx))\ (\lambda R \lambda a.(R)\lambda b.Lab)\lambda B.\exists y\ ((\lambda v.Mv)y \wedge By) \\
&= (\lambda Q.\forall x\ (Wx \to Qx))\ (\lambda R \lambda a.(R)\lambda b.Lab)\lambda B.\exists y\ (My \wedge By) \\
&= (\lambda Q.\forall x\ (Wx \to Qx))\ \lambda a.(\lambda B.\exists y\ (My \wedge By))\lambda b.Lab \\
&= (\lambda Q.\forall x\ (Wx \to Qx))\ \lambda a.\exists y\ (My \wedge (\lambda b.Lab)y) \\
&= (\lambda Q.\forall x\ (Wx \to Qx))\ \lambda a.\exists y\ (My \wedge Lay) \\
&= \forall x\ (Wx \to (\lambda a.\exists y\ (My \wedge Lay))x) \\
&= \forall x\ (Wx \to \exists y\ (My \wedge Lxy))
\end{aligned}
$$

Current stage

S	$\rightarrow$	NP	VP
$[\![S]\!]$	$\leftarrow$	$([\![NP]\!])$	$[\![VP]\!]$
0	$\leftarrow$	(1)	2
NP	$\rightarrow$	PN	
0	$\leftarrow$	$(\Phi)1$	
VP	$\rightarrow$	IV	
0	$\leftarrow$	1	
NP	$\rightarrow$	Det	N
0	$\leftarrow$	(1)	2
VP	$\rightarrow$	TV	NP
0	$\leftarrow$	$((\Psi)1)$	2

PN	$\rightarrow$	Sam
0	$\leftarrow$	s
Φ	$=$	$\lambda u \lambda P.Pu$
IV	$\rightarrow$	sleeps
0	$\leftarrow$	$\lambda x.Sx$
TV	$\rightarrow$	loves
0	$\leftarrow$	$\lambda x \lambda y.Lxy$
Ψ	$=$	$\lambda R \lambda P \lambda x.(P)(R)x$
Det	$\rightarrow$	a
0	$\leftarrow$	$\lambda P \lambda Q.\exists x\,(Px \wedge Qx)$
Det	$\rightarrow$	every
0	$\leftarrow$	$\lambda P \lambda Q.\forall x\,(Px \rightarrow Qx)$
N	$\rightarrow$	man
0	$\leftarrow$	$\lambda x.Mx$