Formal Languages and Linguistics

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General introduction

1. Mathematicians (incl. Chomsky) have formalized the notion of **language**

   oversimplification ?
   maybe...

2. It buys us:
   2.1 Tools to think about theoretical issues about language/s (expressiveness, complexity, comparability...)
   2.2 Tools to manipulate concretely language (e.g. with computers)
   2.3 A research programme:
      - Represent the syntax of natural language in a fully unambiguously specified way

Now let's get familiar with the mathematical notion of language

Overview

Alphabet, word

Def. 1 (Alphabet)

An *alphabet* Σ is a finite set of symbols (letters).
The *size* of the alphabet is the cardinal of the set.

Def. 2 (Word)

A *word* on the alphabet Σ is a finite sequence of letters from Σ.
Formally, let $[p] = (1, 2, 3, 4, ..., p)$ (ordered integer sequence).
Then a word is a *mapping*

$$u : [p] \longrightarrow \Sigma$$

p, the length of u, is noted $|u|$.

Examples I

Alphabet {·, −}

Words −−−·

·

··−·

. . .

Alphabet {·− , −··· , −·−· , −·· , · , . . . }

Words ··· −−− ···

−·· ·· −·· · ·−· −−− −

·− −··· −··· ·− ·− −··· −··· ·−

. . .

Examples II

Alphabet	$\{0,1,2,3,4,5,6,7,8,9,\cdot\}$
Words	$235\cdot29$
	$007\cdot12$
	$\cdot1\cdot1\cdot00\cdot\cdot$
	~~$3\cdot1415962\ldots$~~ (π)
	...
Alphabet	{a, woman, loves, man }
Words	a
	a woman loves a woman
	man man a loves woman loves a
	...

Monoid

Def. 3 (Σ^*)

Let Σ be an alphabet.
The set of all the words that can be formed with any number of letters from Σ is noted Σ^*

Σ^* includes a word with no letter, noted ε

Example:
$$\Sigma = \{a, b, c\}$$
$$\Sigma^* = \{\varepsilon, a, b, c, aa, ab, ac, ba, \ldots, bbb, \ldots\}$$

N.B.: Σ^* is always infinite, except. . .

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N.B.: Σ^* is always infinite, except…
if $\Sigma = \emptyset$. Then $\Sigma^* = \{\varepsilon\}$.

Structure of Σ^*

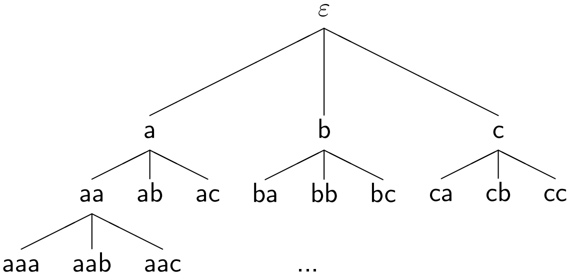
Let k be the size of the alphabet $k = |\Sigma|$.

Then Σ^* contains :

$k^0 = 1$	word(s) of 0 letters (ε)
$k^1 = k$	word(s) of 1 letters
k^2	word(s) of 2 letters
$\ldots$	
k^n	words of n letters, $\forall n \geq 0$

Representation of Σ^*

$\Sigma = \{a, b, c\}$



- Words can be enumerated according to different orders
- Σ^* is a *countable* set

Concatenation

Σ^* can be equipped with a binary operation: *concatenation*

Def. 4 (Concatenation)

Let $[p] \xrightarrow{u} \Sigma$, $[q] \xrightarrow{w} \Sigma$. The concatenation of u and w, noted uw ($u.w$) is thus defined:

$$uw: \quad [p+q] \longrightarrow \Sigma$$
$$uw_i = \begin{cases} u_i & \text{for} \quad i \in [1, p] \\ w_{i-p} & \text{for} \quad i \in [p+1, p+q] \end{cases}$$

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Example : u bacba
v cca

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Example :		
	u	bacba
	v	cca
	uv	bacbacca

Factor

Def. 5 (Factor)

A *factor* w of u is a subset of adjacent letters in u.

$-w$ is a factor of u $\Leftrightarrow$ $\exists u_1, u_2$ s.t. $u = u_1 w u_2$
$-w$ is a left factor (*prefix*) of u $\Leftrightarrow$ $\exists u_2$ s.t. $u = w u_2$
$-w$ is a right factor (*suffix*) of u $\Leftrightarrow$ $\exists u_1$ s.t. $u = u_1 w$

Def. 6 (Factorization)

We call *factorization* the decomposition of a word into factors.

Role of concatenation

1. Words have been defined on Σ.
   If one takes two such words, it's always possible to form a new word by concatenating them.
2. Any word can be factorised in many different ways:
   $a\,b\,a\,c\,c\,a\,b$

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3. Since all letters of Σ form a word of length 1 (this set of words is called the *base*),
4. any word of Σ^* can be seen as a (unique) sequence of concatenations of length 1 words :
   $a\,b\,a\,c\,c\,a\,b$
   $((((((ab)a)c)c)a)b)$
   $((((((a.b).a).c).c).a).b)$

Properties of concatenation

1. Concatenation is non commutative
2. Concatenation is associative
3. Concatenation has an identity (neutral) element: ε

Notation : $a.a.a = a^3$

1. $uv.w \neq w.uv$
2. $(u.v).w = u.(v.w)$
3. $u.\varepsilon = \varepsilon.u = u$

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or, equivalently,
A language on Σ is a subset of Σ^*

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$L_2 = \{a, aa, aaa, aaaa \ldots\}$ or $L_2 = \{a^i \ / \ i \geq 1\}$	infinite language

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$L_5 = \Sigma^*$	

Examples II

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Let $\Sigma' = \{\text{a, man, who, saw, fell}\}$.

$$L' = \left\{ \begin{array}{l} \text{a man fell,} \\ \text{a man who saw a man fell,} \\ \text{a man who saw a man who saw a man fell,} \\ \dots \end{array} \right\}$$

Set operations

Since a language is a set, usual set operations can be defined:

- union
- intersection
- set difference

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⇒ One may describe a (complex) language as the result of set operations on (~~simpler~~) languages:
$\{a^{2k} \ / \ k \geqslant 1\} = \{a, aa, aaa, aaaa, \ldots\} \cap \{ww \ / \ w \in \Sigma^*\}$

Additional operations

Def. 8 (product operation on languages)

One can define the *language product* and its closure *the Kleene star* operation:

- The *product* of languages is thus defined:

$$L_1.L_2 \quad = \quad \{uv \,/\, u \in L_1 \;\&\; v \in L_2\}$$

  Notation: $\overbrace{L.L.L \ldots L}^{k \text{ times}} = L^k \; ; \; L^0 = \{\varepsilon\}$

- The Kleene star of a language is thus defined:

$$L^* = \bigcup_{n \geqslant 0} L^n$$

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