Formal Languages and Linguistics

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General introduction

1. Mathematicians (incl. Chomsky) have formalized the notion of **language**

   It might be thought of as an oversimplification,
   always the same story...

2. It buys us:
   2.1 Tools to think about theoretical issues about language/s (expressiveness, complexity, comparability...)
   2.2 Tools to manipulate concretely language (e.g. with computers)
   2.3 A research programme:
      - Represent the syntax of natural language in a fully unambiguously specified way

Now let's get familiar with the mathematical notion of language

Overview

Alphabet, word

Def. 1 (Alphabet)

An *alphabet* Σ is a finite set of symbols (letters).
The *size* of the alphabet is the cardinal of the set.

Def. 2 (Word)

A *word* on the alphabet Σ is a finite sequence of letters from Σ.
Formally, let $[p] = (1, 2, 3, 4, ..., p)$ (ordered integer sequence).
Then a word is a *mapping*

$$u : [p] \longrightarrow \Sigma$$

p, the length of u, is noted $|u|$.

Examples I

Alphabet {·, —}

Words

— — — ·

·

· · — ·

. . .

Alphabet {·— , —··· , —·—· , —·· , · , . . . }

Words

··· ——— ···

—·· ·· —·· · ·—· ——— —

·— —··· —··· ·— ·— —··· —··· ·—

. . .

Examples II

Alphabet	$\{0,1,2,3,4,5,6,7,8,9,\cdot\}$
Words	$235 \cdot 29$
	$007 \cdot 12$
	$\cdot 1 \cdot 1 \cdot 00 \cdot \cdot$
	$\cancel{3 \cdot 1415962 \ldots}$ (π)
	...
Alphabet	{a, woman, loves, man }
Words	a
	a woman loves a woman
	man man a loves woman loves a
	...

Monoid

Def. 3 (Σ^*)

Let Σ be an alphabet.
The set of all the words that can be formed with any number of letters from Σ is noted Σ^*

Σ^* includes a word with no letter, noted ε

Example:
$$\begin{aligned} \Sigma &= \{a, b, c\} \\ \Sigma^* &= \{\varepsilon, a, b, c, aa, ab, ac, ba, \ldots, bbb, \ldots\} \end{aligned}$$

N.B.: Σ^* is always infinite, except...

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N.B.: Σ^* is always infinite, except. . .
if $\Sigma = \emptyset$. Then $\Sigma^* = \{\varepsilon\}$.

Structure of Σ^*

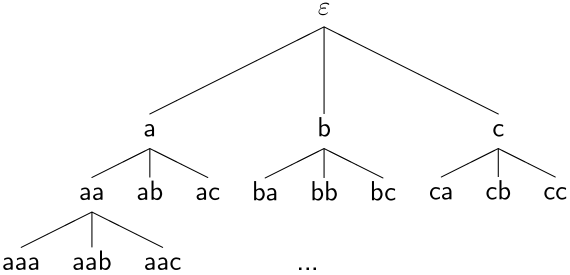
Let k be the size of the alphabet $k = |\Sigma|$.

Then Σ^* contains :

$k^0 = 1$	word(s) of 0 letters (ε)
$k^1 = k$	word(s) of 1 letters
k^2	word(s) of 2 letters
...	
k^n	words of n letters, $\forall n \geq 0$

Representation of Σ^*

$$\Sigma = \{a, b, c\}$$



- Words can be enumerated according to different orders
- Σ^* is a *countable* set

Concatenation

Σ^* can be equipped with a binary operation: *concatenation*

Def. 4 (Concatenation)

Let $[p] \xrightarrow{u} \Sigma$, $[q] \xrightarrow{w} \Sigma$. The concatenation of u and w, noted uw ($u.w$) is thus defined:

$$uw : \quad [p+q] \longrightarrow \Sigma$$
$$uw_i = \begin{cases} u_i & \text{for} \quad i \in [1, p] \\ w_{i-p} & \text{for} \quad i \in [p+1, p+q] \end{cases}$$

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	v	cca

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Factor

Def. 5 (Factor)

A *factor* w of u is a subset of adjacent letters in u.

$-w$ is a factor of u $\Leftrightarrow$ $\exists u_1, u_2$ s.t. $u = u_1 w u_2$
$-w$ is a left factor (*prefix*) of u $\Leftrightarrow$ $\exists u_2$ s.t. $u = w u_2$
$-w$ is a right factor (*suffix*) of u $\Leftrightarrow$ $\exists u_1$ s.t. $u = u_1 w$

Def. 6 (Factorization)

We call *factorization* the decomposition of a word into factors.

Role of concatenation

1. Words have been defined on Σ.
   If one takes two such words, it's always possible to form a new word by concatenating them.
2. Any word can be factorised in many different ways:
   $a\,b\,a\,c\,c\,a\,b$

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   $(a)(b)(a)(c)(c)(a)(b)$
3. Since all letters of Σ form a word of length 1
   (this set of words is called the *base*),
4. any word of Σ^* can be seen as a (unique) sequence of concatenations of length 1 words :
   $a\,b\,a\,c\,c\,a\,b$
   $((((((ab)a)c)c)a)b)$
   $((((((a.b).a).c).c).a).b)$

Properties of concatenation

1. Concatenation is non commutative
2. Concatenation is associative
3. Concatenation has an identity (neutral) element: ε

1. $uv.w \neq w.uv$
2. $(u.v).w = u.(v.w)$
3. $u.\varepsilon = \varepsilon.u = u$

Notation : $a.a.a = a^3$

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or, equivalently,
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$L_5 = \Sigma^*$	

Examples II

Let $\Sigma = \{\text{a}, \text{man}, \text{loves}, \text{woman}\}$.

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Let $\Sigma' = \{\text{a, man, who, saw, fell}\}$.

$$L' = \left\{ \begin{array}{l} \text{a man fell,} \\ \text{a man who saw a man fell,} \\ \text{a man who saw a man who saw a man fell,} \\ \ldots \end{array} \right\}$$

Set operations

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- union
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$\Rightarrow$ One may describe a (complex) language as the result of set operations on (~~simpler~~) languages:
$\{a^{2k} \;/\; k \geqslant 1\} = \{a, aa, aaa, aaaa, \ldots\} \cap \{ww \;/\; w \in \Sigma^*\}$

Additional operations

Def. 8 (product operation on languages)

One can define the *language product* and its closure *the Kleene star* operation:

- The *product* of languages is thus defined:
$$L_1.L_2 \quad = \quad \{uv \,/\, u \in L_1 \;\&\; v \in L_2\}$$
Notation: $\overbrace{L.L.L\ldots L}^{k \text{ times}} = L^k \; ; \; L^0 = \{\varepsilon\}$
- The Kleene star of a language is thus defined:
$$L^* = \bigcup_{n \geqslant 0} L^n$$

Regular expressions

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$$(\{a\} \cup \{b\})^*.\{c\} = \{c, ac, abc, bc, \ldots, baabaac, \ldots\}$$

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... but not all languages can be thus characterized.

Overview

Back to "Natural" Languages

English as a formal language:

- **alphabet:** morphemes (often simplified to words —depending on your view on flexional morphology)
  - ⇒ Finite at a time t by hypothesis
- **words:** well formed English sentences
  - ⇒ English sentences are all finite by hypothesis
- **language:** English, as a set of an infinite number of well formed combinations of "letters" from the alphabet

Discussion I

1. is the alphabet finite?

   closed class morphemes obviously

   open class morphemes what about "new words"?

   - **morphological derivations** can be seen as produced from an unchanged inventory (1)
   - **other words**
     - loan words (rare)
     - lexical inventions (rare)
     - change of category (2) (bounded)

   $\Rightarrow$ negligable

(1) motherese = mother+ese

Discussion II

(2) $\text{american}_A \rightarrow \text{american}_N$

2. is English infinite ?
   - It is supposed that you can always profer a longer sentence than the previous one by adding linguistic material preserving well-formedness.
   - Compatible with the working memory limit

     (Langendoen & Postal, 1984)
3. is language discrete ?
   Well, that's another story

About infinity

Linguists sometimes have trouble with infinity:

> In order for there to be an infinite number of sentences in a language there must either be an infinite number of words in the language (clearly not true) or there must be the possibility of infinite length sentences. The product of two finite numbers is always a finite number. (Mannell, 1999)
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von Humbolt: *language is an infinite use of finite means*

(quoted by Chomsky)

Good questions

Why would one consider natural language as a formal language?

- it allows to $\boxed{\text{describe}}$ the language in a formal/compact/elegant way
- it allows to $\boxed{\text{compare}}$ various languages (via classes of languages established by mathematicians)
- it give algorithmic tools to $\boxed{\text{recognize}}$ and to $\boxed{\text{analyse}}$ words of a language.

recognize u	: decide whether $u \in L$
analyse u	: show the internal structure of u

Overview

Definition

3 possible definitions

1. a regular language can be generated by a regular grammar
2. a regular language can be defined by rational expressions
3. a regular language can be recognized by a finite automaton

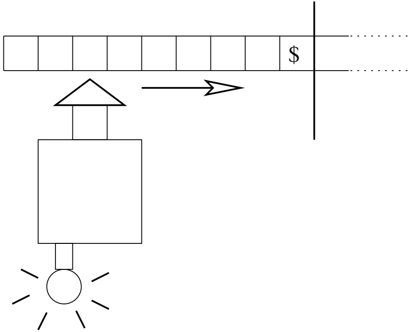
Def. 9 (Rational Language)

A rational language on Σ is a subset of Σ^* inductively defined thus:

- $\emptyset$ and $\{\varepsilon\}$ are rational languages ;
- for all $a \in X$, the singleton $\{a\}$ is a rational language ;
- for all g and h rational, the sets $g \cup h$, $g.h$ and g^* are rational languages.

Overview

Metaphoric definition



Formal definition

Def. 10 (Finite deterministic automaton (FDA))

A finite state deterministic automaton $\mathcal{A}$ is defined by :

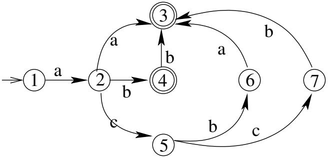
$$\mathcal{A} = \langle Q, \Sigma, q_0, F, \delta \rangle$$

- Q is a finite set of states
- Σ is an alphabet
- q_0 is a distinguished state, the initial state,
- F is a subset of Q, whose members are called final/terminal states
- δ is a mapping **fonction** from $Q \times \Sigma$ to Q. Notation $\delta(q, a) = r$.

Example

Let us consider the (finite) language $\{aa, ab, abb, acba, accb\}$.
The following automaton recognizes this langage: $\langle Q, \Sigma, q_0, F, \delta\rangle$, avec $Q = \{1,2,3,4,5,6,7\}$, $\Sigma = \{a, b, c\}$, $q_0 = 1$, $F = \{3,4\}$, and δ is thus defined:

$$
\delta : \quad
\begin{array}{l}
(1,\text{a}) \mapsto 2 \\
(2,\text{a}) \mapsto 3 \\
(2,\text{b}) \mapsto 4 \\
(2,\text{c}) \mapsto 5 \\
(4,\text{b}) \mapsto 3 \\
(5,\text{b}) \mapsto 6 \\
(5,\text{c}) \mapsto 7 \\
(6,\text{a}) \mapsto 3 \\
(7,\text{b}) \mapsto 3
\end{array}
$$



	a	b	c
$\rightarrow$ 1	2		
2	3	4	5
$\leftarrow$ 3			
$\leftarrow$ 4		3	
5		6	7
6	3		
7		3	

Recognition

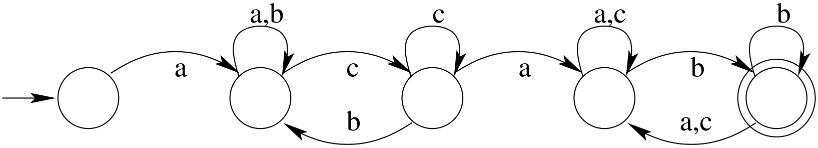
Recognition is defined as the existence of a sequence of states defined in the following way. Such a sequence is called a path in the automaton.

Def. 11 (Recognition)

A word $a_1a_2...a_n$ is **recognized/accepted** by an automaton iff there exists a sequence $k_0, k_1, ..., k_n$ of states such that:

$$k_0 = q_0$$
$$k_n \in F$$
$$\forall i \in [1, n], \quad \delta(k_{i-1}, a_i) = k_i$$

Example



Exercices

Let $\Sigma = \{a, b, c\}$. Give deterministic finite state automata that accept the following languages:

1. The set of words with an even length.
2. The set of words where the number of occurrences of b is divisible by 3.
3. The set of words ending with a b.
4. The set of words not ending with a b.
5. The set of words non empty not ending with a b.
6. The set of words comprising at least a b.
7. The set of words comprising at most a b.
8. The set of words comprising exactly one b.