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2. Syntax

(a) First Order Logic Language

(i) If A is a predicate constant, of arity n, and each $t_1...t_n$ an individual constant or variable, then $A(t_1, ..., t_n)$ is a wff.
(ii) If φ is a wff, then so is $\neg\varphi$.
(iii) If φ and ψ are wffs, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.
(iv) If φ is a wff and x a variable, then $\forall x\varphi$ and $\exists x\varphi$ are wffs.
(v) Nothing else is a wff.

(b) Scope

If $\forall x\psi$ is a sub-formula of φ, then ψ is called the **scope** of this occurrence of the quantifier $\forall x$ in φ. Same definition for $\exists x$.

(c) Bound/Free variable

(a) An occurrence of a variable x in the formula ϕ (which is not part of a quantifer) is called **free** if this occurrence of x is not in the scope of a quantifier $\forall x$ ou $\exists x$ occurring in ϕ.
(b) If $\forall x\psi$ (or $\exists x\psi$) is a sub-formula of ϕ and x is free in ψ, then this occurrence of x is called **bound** by the quantifier $\forall x$ (or $\exists x$).

(d) Sentence vs wff

A **sentence** is a formula with no free variable.

3. Semantics

(c) Tarskian truth definition

Let $[\![\alpha]\!]^g_{\mathcal{M}}$ be the denotation of α in the model $\mathcal{M} = \langle D, I\rangle$ and with the assignment g.

$[\![t]\!]^g_{\mathcal{M}} = I(t)$ if t is an individual constant
$[\![t]\!]^g_{\mathcal{M}} = g(t)$ if t is a variable

$$[\![P(t_1, ...t_n)]\!]^g_{\mathcal{M}} = 1 \text{ iff } \langle [\![t_1]\!]^g_{\mathcal{M}}, ...[\![t_n]\!]^g_{\mathcal{M}}\rangle \in I(P).$$

If φ and ψ are wfss,

$$\begin{array}{lll} [\![\neg\varphi]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 0 \\ [\![(\varphi \wedge \psi)]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 1 \text{ and } [\![\psi]\!]^g_{\mathcal{M}} = 1 \\ [\![(\varphi \vee \psi)]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 1 \text{ or } [\![\psi]\!]^g_{\mathcal{M}} = 1 \\ [\![(\varphi \rightarrow \psi)]\!]^g_{\mathcal{M}} = 1 & \text{iff} & [\![\varphi]\!]^g_{\mathcal{M}} = 0 \text{ or } [\![\psi]\!]^g_{\mathcal{M}} = 1 \end{array}$$

$$\boxed{[\![\exists y\ \varphi]\!]^g_{\mathcal{M}} = 1 \text{ iff there is a } d \in D \text{ s.t. } [\![\varphi]\!]^{g[y/d]}_{\mathcal{M}} = 1}$$

similarly,

$$\boxed{[\![\forall y\ \varphi]\!]^g_{\mathcal{M}} = 1 \text{ iff for all } d \in D,\ [\![\varphi]\!]^{g[y/d]}_{\mathcal{M}} = 1}$$

Il φ is a sentence :

$$\underline{[\![\varphi]\!]_{\mathcal{M}} = 1 \text{ iff there is an assignment } g \text{ such that } [\![\varphi]\!]^g_{\mathcal{M}} = 1}$$

4. Results

Equivalences

- Bound variables are "dummy" : their name no longer matters.

$$\forall x\ Fx \quad \equiv \quad \forall y\ Fy$$

But beware of unintended captures :

$$\forall x\ (Fx \wedge Gy) \quad \not\equiv \quad \forall y\ (Fy \wedge Gy)$$

- Duality rules (*de Morgan* laws)

$$\forall x\ \alpha \quad \equiv \quad \neg\exists\ \neg\alpha$$

for instance :

$$\forall x\ Rx \quad \equiv \quad \neg\exists\ \neg Rx$$

All is relative $\approx$ *Nothing is absolute* ($\approx$ *non relative)*

$$\forall x\ (Px \rightarrow Kx) \quad \equiv \quad \neg\exists x\ (Px \wedge \neg Kx)$$

All professors are kind $\approx$ *There are no non-kind professors*

Other variants :

$$\exists x\ \alpha \quad \equiv \quad \neg\forall x\ \neg\alpha$$

$$\neg\exists x\ \alpha \quad \equiv \quad \forall x\ \neg\alpha$$

$$\neg\forall x\ \alpha \quad \equiv \quad \exists x\ \neg\alpha$$

- Distribution rules :

$$\forall x\ (\alpha \wedge \beta) \quad \equiv \quad (\forall x\ \alpha \wedge \forall x\ \beta)$$

All is rare and expensive $\approx$ *All is rare and all is expensive*

But :

$$\forall x\ (\alpha \vee \beta) \quad \not\equiv \quad (\forall x\ \alpha \vee \forall x\ \beta)$$

All is either relative or absolute $\not\approx$ *Either all is relative or all is absolute*

$$\exists x\ (\alpha \vee \beta) \quad \equiv \quad (\exists x\ \alpha \vee \exists x\ \beta)$$

But :

$$\exists x\ (\alpha \wedge \beta) \quad \not\equiv \quad (\exists x\ \alpha \wedge \exists x\ \beta)$$

$$\exists x\ (\alpha \rightarrow \beta) \quad \equiv \quad (\forall x\ \alpha \rightarrow \exists x\ \beta)$$

- Conditional distribution ($\bar{\beta}$ doesn't contain free occurrences of x)

$$\bar{\beta} \quad \equiv \quad \forall x \bar{\beta}$$

$$\bar{\beta} \quad \equiv \quad \exists x \bar{\beta}$$

$$\forall x\ (\alpha \vee \bar{\beta}) \quad \equiv \quad (\forall x\ \alpha \vee \bar{\beta})$$

$$\exists x\ (\alpha \wedge \bar{\beta}) \quad \equiv \quad \exists x\ \alpha \wedge \bar{\beta}$$

$$\forall x\ (\alpha \rightarrow \bar{\beta}) \quad \equiv \quad \exists x\ \alpha \rightarrow \bar{\beta}$$

Every entity is such that if it breaks, there is noise $\approx$ *If some entity breaks, there is noise*

$$\forall x\ (\bar{\beta} \rightarrow \alpha) \quad \equiv \quad \bar{\beta} \rightarrow \forall x\ \alpha$$

For all person, if there is noise, s/he is upset $\approx$ *If there is noise, everyone is upset*